

(Yet another) cyclic quantum hyperbolic invariant

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Workshop on Hyperbolic Structures and Quantum Invariants

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Thank you

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- and for organizing such a wonderful conference

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 - please correct me
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Please

- interrupt with questions
 - I would rather have part of my talk be understood than say all of it

Overview of result

- K framed, oriented knot in S^3
- $\rho : \pi_1(S^3 \setminus K) \rightarrow \mathrm{SL}_2(\mathbb{C})$ representation
 - plus choice of eigenline for peripheral subgroup (**decoration** or **framing**)
- μ logarithm of the distinguished meridian eigenvalue

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- μ logarithm of the distinguished meridian eigenvalue

Theorem [Me, parts joint with N. Reshetikhin] For any knot K in S^3 we have a sequence of cyclic quantum hyperbolic invariants

$$\mathcal{Z}_N^\psi(K, \rho, \mu) \in \mathbb{C}, N \geq 2$$

What is $\mathcal{Z}_N^\psi$?

geometric extensions of the Kashaev invariant

- $\mathcal{Z}_N^\psi(K, (-1)^{N-1}, \frac{N-1}{2})$ recovers Kashaev invariant at trivial representation

quantization of the $\mathrm{SL}_2(\mathbb{C})$ Chern-Simons invariant

- in an algebraic/formal sense
- line bundles $\rightarrow$ vector bundles
- dilogarithms $\rightarrow$ quantum dilogarithms

My goals in this talk are to

1. explain the quantization analogy
2. convince you that it is plausible and interesting.

You might be thinking:

Other people have done this before:

- [Baseilhac and Benedetti, 2004] (matrix dilogarithms)
- [Bonahon and Liu, 2007] (quantum Teichmüller theory)
- [Bonahon, Wong, and Yang, 2021] (skein algebras)

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Why should I care about a new version?

HOW STANDARDS PROLIFERATE:
(SEE: A/C CHARGERS, CHARACTER ENCODINGS, INSTANT MESSAGING, ETC.)



My goals in this talk are to

1. explain the quantization analogy
2. convince you that it is plausible and interesting.
3. suggest how these ideas might be helpful for other versions of QHI

Plan of the talk

Background What is a cyclic quantum hyperbolic invariant?

Classical version What is the complex Chern-Simons invariant?

Quantum version How does this lead to $\mathcal{Z}_N^\psi$?

Future directions What ideas might be helpful for the other 14 3 constructions?

- K is a knot in S^3 (frequently hyperbolic) (other versions work for 3-manifolds)
- $\mathcal{X}_K$ is some version of the $\mathrm{SL}_2(\mathbb{C})$ character variety of K (ie the moduli space of hyperbolic structures on $S^3 \setminus K$)

Definition: A **cyclic quantum hyperbolic invariant** is a family of functions

$$Z_N(K) : \mathcal{X}_K \rightarrow \mathbb{C}, \quad N \in \mathbb{Z}, N \geq 1$$

that are a geometric deformation of quantum invariants. For example,

$$\begin{aligned} Z_N(K, 1) &= \text{Kashaev's invariant } \mathcal{K}_N(K) \\ &= N\text{th colored Jones polynomial} \\ &\quad \text{at } q \rightarrow e^{2\pi i/N} \end{aligned}$$

Other typical features:

- $Z_N(K, \rho)$ is a discrete state sum
 - tensor coefficients involve **quantum dilogarithms**
- $Z_N(K, \rho)$ is related to some of
 - quantum Chern-Simons theory for $\mathrm{SL}_2(\mathbb{C})$
 - ordinary WRT uses a compact group like $\mathrm{SU}(2)$
 - quantizations of the space $\mathcal{X}_K$ (“quantum Teichmüller theory”)

Another feature: expected to satisfy a **Volume Conjecture**:

$$Z_N(K, \rho) \sim N^{\frac{3}{2}} \tau e^{N\varsigma(K)} \text{ as } N \rightarrow \infty$$

- $\varsigma(K)$ complex Chern-Simons invariant (real part hyperbolic volume)
- τ related to adjoint hyperbolic torsion
- $N \rightarrow \infty$ is semiclassical limit in physics language

The surprising part is that this holds for *many* ρ , not just the geometric representation.

In particular the Kashaev-Murakami-Murakami conjecture is about the case where ρ is trivial.

The hope is that we can use these geometric invariants to

1. understand and prove the Volume Conjecture
2. get more information about topology and geometry

Why “cyclic”?

The invariants of

- [Baseilhac and Benedetti, 2004]
- [Bonahon and Liu, 2007]
- [Bonahon, Wong, and Yang, 2021]

have

- cyclic dilogarithms (of some form)
- finite, discrete state sums
- expect positive growth in semiclassical limit

I call these **cyclic** QHI.

The invariants of

- [Andersen and Kashaev, 2014]
- [Kashaev, Luo, and Vartanov, 2015]
- [Liu, Ming, Sun, Wu, and Yang, 2026]

have

- generic parameter, not a root of unity
- noncompact state integrals
 - oriented differently in parameter space
 - expect negative growth

I call these **principal** QHI.

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There are strong analogies but the precise relationship between these is unclear.

I hope that the ideas in this talk will help illuminate it.

The invariants of

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have

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Our approach

Use continuous family of de Concini-Kac modules of $\mathcal{U}_\omega(\mathfrak{sl}_2)$ to do a geometric deformation/twist/extension of colored Jones polynomials.

Get an invariant $\mathcal{Z}_N^\psi(K, \rho, \mu)$ of knots (and links and tangles) depending on geometric structure ρ .

Some nice properties:

- defined for *all* ρ (no non-degeneracy requirement)
- does not depend on choice of diagram
 - need to satisfy some nondegeneracy conditions *but* can always satisfy this via gauge transformations
- no phase ambiguities
- $\mathcal{Z}_2^\psi(K, \rho, \mu) \mathcal{Z}_2^\psi(\overline{K}, \overline{\rho}, \mu) = \tau(S^3 \setminus K, \rho)$
twisted Reidemeister torsion
- for sufficiently nondegenerate ρ admits a state integral presentation

Today I want to focus on the relationship between $\mathcal{Z}_N^\psi$ and the classical complex Chern-Simons invariant

I think this gives new information on what the theory underlying all the QHI is

In addition, the version focusing just on $\mathcal{U}_\omega(\mathfrak{sl}_2)$ is harder to understand! The geometric perspective has made the theory better, so I want to share that.

What is the complex Chern-Simons invariant?

- M closed 3-manifold
- $\rho : \pi_1(M) \rightarrow \mathrm{SL}_2(\mathbb{C})$ (generalized) hyperbolic structure

Choose a “connection” $A \in \Omega^1(M, \mathfrak{sl}_2)$ whose holonomy is ρ . The **Chern-Simons invariant** is

$$S(A) := -\frac{i}{4} \int_M \mathrm{tr}[A, dA] + \frac{2}{3} \mathrm{tr}[A, A \wedge A]$$

Theorem This is independent of the gauge class of A modulo $2\pi i$ so we get an invariant

$$\mathcal{I}(M, \rho) = e^{S(A)}$$

Theorem [Yoshida, 1985] $\log \mathcal{I}(M, \rho)$ is

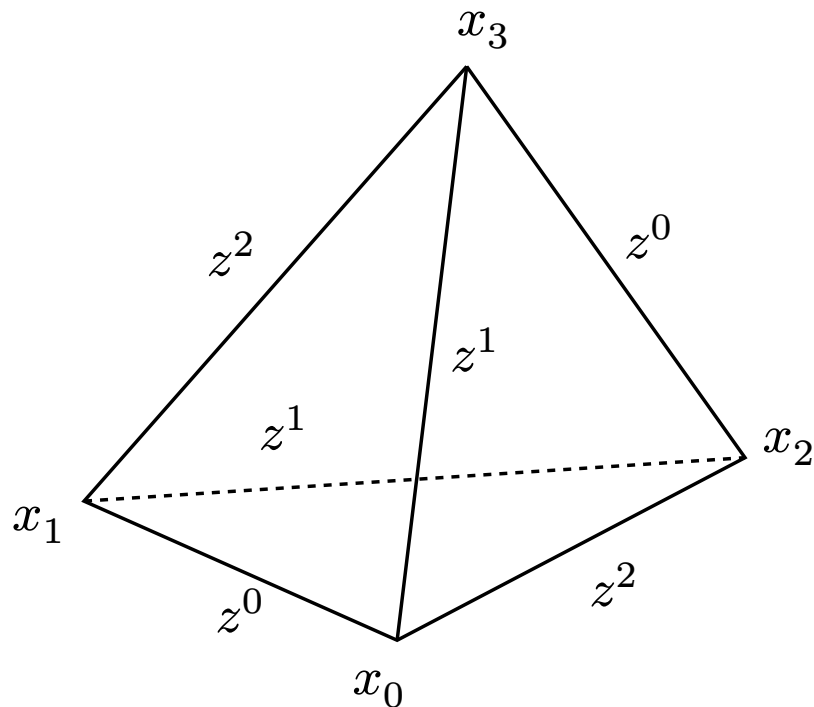
$$\frac{1}{2\pi} \mathrm{Vol}(M, g) + i \mathrm{CS}_{\mathrm{SO}(3)}(M, g)$$

- g metric induced by ρ
- imaginary part is CS of frame field of g
- $\log \mathcal{I}$ is thus called **complex volume**

- Computing $\mathcal{I}$ in practice can be quite difficult.
 - for example, one might want to use the connection A coming from the hyperbolic structure of a shaped ideal triangulation
 - how can we find A in practice?
- We might try to compute it by triangulating and working piece-by-piece
- When we cut our closed manifold M into pieces they have boundary
- We need to understand the boundary conditions
 - [Neumann, 2004] explained how to do this using **flattenings**

For a single ideal tetrahedron, geometric data is **shape parameter**

$$z \in \mathbb{C} \setminus \{0, 1\} = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$$



$$z = \text{cross ratio of } x_0, \dots, x_3 \in \mathbb{CP}^1 = \partial\mathbb{H}^3$$

Actually this depends on a choice of edge pair (quad type)

Other pairs get shapes

$$z^1 = \frac{1}{1-z}, z^2 = 1 - \frac{1}{z}$$

Pick a cross-ratio convention that makes my figure correct.

CS depends on additional choice of **flattening** ζ^0, ζ^1 with

$$e^{2\pi i \zeta^0} = z \quad \zeta^0 \text{ is preferred branch of } \log z$$

$$e^{2\pi i \zeta^1} = \frac{1}{1-z} \quad \zeta^1 \text{ is preferred branch of } -\log(1-z) \equiv \log z^1$$

This is a boundary condition [Marché, 2012] for the flat connection. We can compute

$$\mathcal{I}(\zeta^0, \zeta^1) = \exp\left(\mathrm{Li}_2(z) + \zeta^0 \log(1-z) + \frac{1}{2}\zeta^0\zeta^1\right) \text{ where } z = e^{2\pi i \zeta}$$

in terms of **dilogarithm** Li_2 . This is a holomorphic function satisfying a **five-term relation** corresponding to the Pachner 2-3 move on tetrahedra.

Because of branches

$$\mathcal{I}(z, \zeta^0 + 1, \zeta^1) = e^{-\pi i \zeta^1} \mathcal{I}(z, \zeta^0, \zeta^1)$$

$$\mathcal{I}(z, \zeta^0, \zeta^1 + 1) = e^{\pi i \zeta^0} \mathcal{I}(z, \zeta^0, \zeta^1)$$

One perspective

domain is a covering space $\Sigma \rightarrow \mathbb{C} \setminus \{0, 1\}$

Another perspective

$\mathcal{I}$ takes values in a line bundle over
geometric parameter space

What do I mean by “line bundle”?

$$f(z) = z^{\frac{1}{2}} = \exp\left(\frac{1}{2} \log z\right)$$

is not a holomorphic function $\mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$. To fix:

Adjust domain

Take branch cut along $(-\infty, 0]$ and view as a function on Riemann surface over $\mathbb{C} \setminus \{0\}$

Adjust codomain

Take values in a line bundle $\mathcal{L} \rightarrow \mathbb{C} \setminus \{0\}$

What do I mean by “line bundle”?

Fiber over z is $\{(w, \zeta) \mid e^{2\pi i \zeta} = z\}$ modulo relation

$$(w, \zeta + 1) = (-w, \zeta)$$

Each is a one-dimensional vector space,
combine into a line bundle $\mathcal{L} \rightarrow \mathbb{C} \setminus \{0\}$

The map

$$f(z) = \left(e^{\frac{1}{2} \log(z)}, \frac{1}{2\pi i} \log(z) \right)$$

extends to a global section of $\mathcal{L}$

The line bundle for tetrahedra

The moduli space of (oriented, vertex-ordered) ideal tetrahedra is space of shape parameters:

$$\mathcal{X} = \mathbb{C} \setminus \{0, 1\} = \mathbb{C}P^1 \setminus \{0, 1, \infty\}$$

The Chern-Simons invariant of an ideal tetrahedron is a section $\mathcal{I}(\Delta)$ of $\mathcal{L}$:

$(\mathcal{I}(z, \zeta^0, \zeta^1), \zeta^0, \zeta^1)$ is independent of choice of ζ^0, ζ^1

There is a line bundle $\mathcal{L} \rightarrow \mathcal{X}$ whose fiber over z is (w, ζ^0, ζ^1) with relations

$$(w, \zeta^0 + 1, \zeta^1) = (we^{\pi i \zeta^1}, \zeta^0, \zeta^1)$$

$$(w, \zeta^0, \zeta^1 + 1) = (we^{-\pi i \zeta^0}, \zeta^0, \zeta^1)$$

Take a 3-manifold with

- an ideal triangulation $\{T_i\}$
- a hyperbolic structure ρ
- a solution of Thurston's gluing equations encoding ρ
- a **flattening**, which is a coherent choice of flattenings for the T_i
 - satisfies logarithmic version of gluing relation at each edge class
 - not obvious these exist!

Theorem [Neumann, 2004] If you glue tetrahedra together to get a cusped manifold and ρ is boundary-parabolic the dependence on flattenings cancels out and

$$\mathcal{I}(M, \rho) = \prod_i \mathcal{I}(\zeta_i^0, \zeta_i^1)$$

is a complex number.

Works for closed manifolds if you add a Dehn surgery correction term.

Boundary data and normalization

For a knot exterior $M = S^3 \setminus \nu(K)$, a **decoration** of ρ picks an eigenline for the peripheral subgroup, with distinguished meridian and longitude eigenvalues m, ℓ .

Theorem [Kirk and Klassen, 1993] $\mathcal{I}(M, \rho)$ is a section of a line bundle over the decorated character variety.

Choose logarithms μ, λ of m, ℓ . We use the normalization

$$\mathcal{I}^\psi = e^{2\pi i \psi} \mathcal{I} = e^{2\pi i \mu \lambda} \mathcal{I}$$

which removes dependence on the longitude logarithm λ .

Complex Chern-Simons from knot diagrams

I want to use knot diagrams instead of ideal triangulations.

Goal:

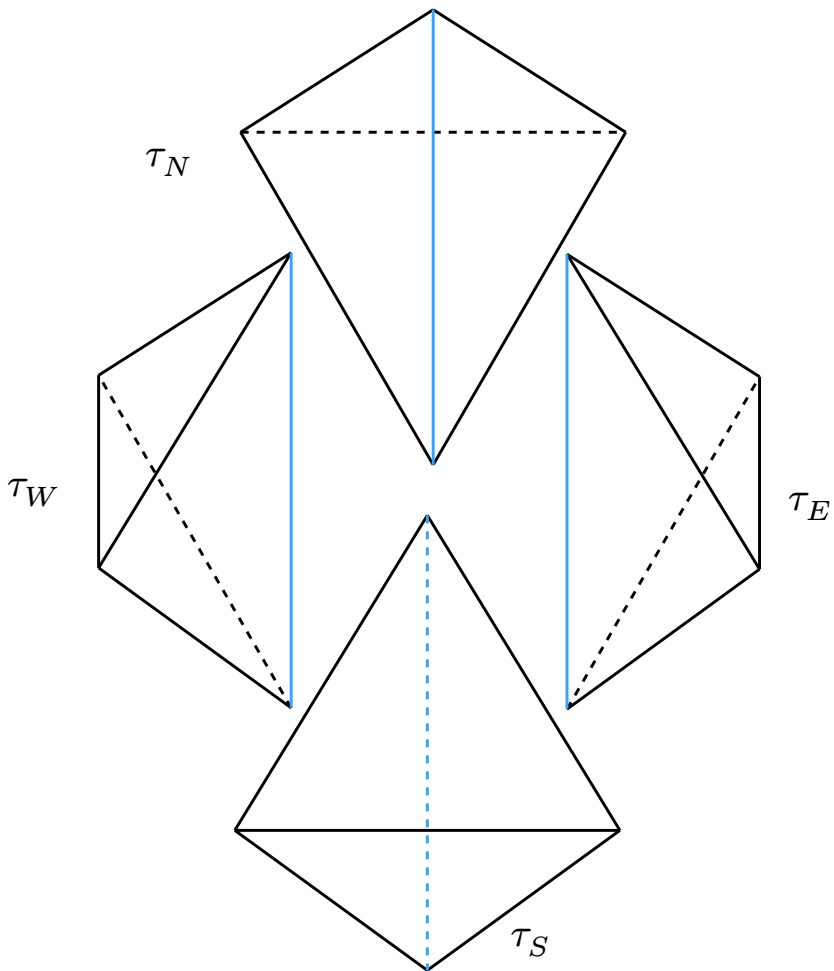
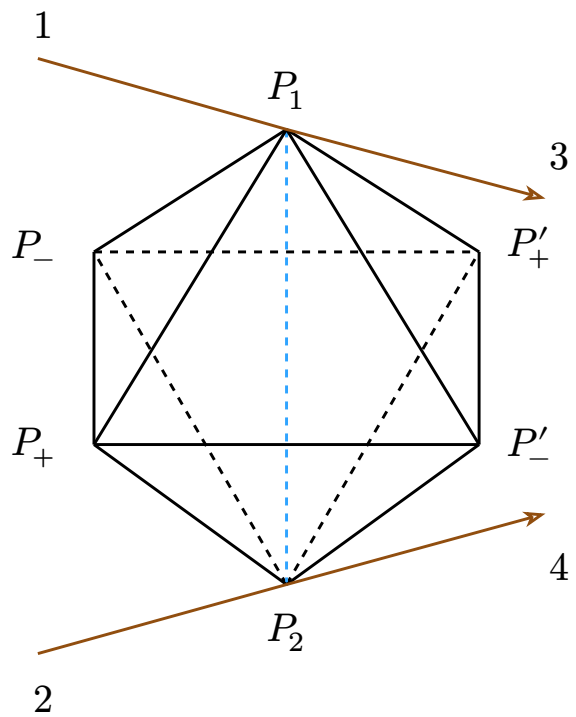
- compute $\mathcal{I}^\psi$ from a knot diagram
- make sense of $\mathcal{I}^\psi$ for tangles

To get there:

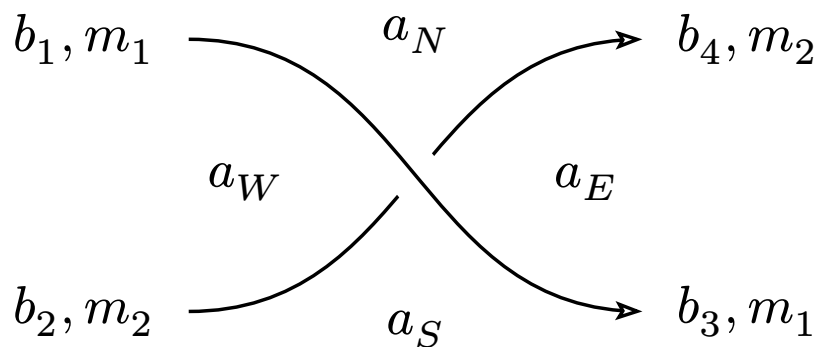
Use **octahedral decompositions**

[D. Thurston, Kashaev, Yokota, Weeks, et al
(?)]

- Each crossing gets an ideal octahedron
- Can split into 4 (or 5) tetrahedra
- Actually semi-ideal (2 material vertices)



We record our decorated representation ρ using the **octahedral coordinates** of [Kim, Kim, and Yoon, 2018]. For a crossing, my variant is



Shape parameters of tetrahedra at corners are of form $\frac{b_j}{b_k} m_i^{\pm 1}$

Write $\mathcal{R} \left(\begin{array}{c} \curvearrowright \end{array} \right)$ for space of these coordinates for a crossing.

Proposition If you assign these to the regions and segments of a diagram and they satisfy relations (corresponding to gluing rels) you get some ρ .

Conversely every ρ arises this way (up to gauge transformation)

If we choose logarithms β_i, α_j, μ_k of these parameters we get a flattening automatically. (Same reasons as for Ptolemy coordinates of [Zickert, 2009])

Can work out (use [Cho, 2018], [Inoue and Kabaya, 2014], [Hikami and Inoue, 2015] as guide)

$$\begin{aligned} \mathcal{I}^\psi \left(\text{diagram} \right) \\ &= \Omega(\beta_1, \beta_2, \beta_3, \beta_4, \mu_1, \mu_2) \\ &:= e^{2\pi i(\dots)} \frac{L(\beta_4 - \beta_1) L(\beta_2 - \beta_3 + \mu_2 - \mu_1)}{L(\beta_2 - \beta_1 - \mu_1) L(\beta_4 - \beta_3 + \mu_2)} \end{aligned}$$

This thing is a section of a line bundle over $\mathcal{R}$ (diagram), with monodromy like

$$\Omega(\beta_1 + 1, \dots) = \frac{a_N}{a_W} \Omega(\beta_1, \dots)$$

Why do the α not appear?

That's why we use $\mathcal{I}^\psi$ and not $\mathcal{I}$!

Corresponds to dilogarithm variant L on left, which drops the $\zeta^0 \zeta^1$ term

What goes on segments?

We want to define a line bundle $\mathcal{I}^\psi \left(\longrightarrow \right)$ over space

$$\mathcal{R} \left(\longrightarrow \right) = \{(a, b, m)\} \text{ of segment parameters}$$

Fibers

Set $\Lambda(b) = \{\beta \mid e^{2\pi i \beta} = b\}$. Quasi-periodic functions on $\Lambda(b)$:

$$\mathcal{I}^\psi(a, b) :=$$

$$\{f : \Lambda(b) \rightarrow \mathbb{C} \mid f(\beta + 1) = af(\beta)\}$$

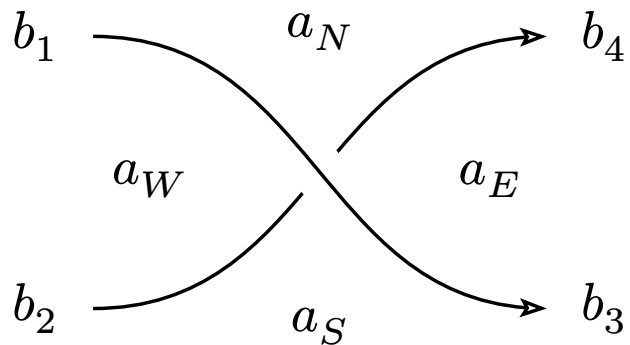
Bases

A logarithm $\beta \in \Lambda(b)$ determines a basis vector

$$\langle \beta \mid : \beta + k \mapsto a^k$$

A choice of branch/logarithm is the **same** as a choice of basis.

Putting it together



is assigned the linear map

$$\mathcal{I}^\psi \left(\frac{a_W}{a_N}, b_1 \right) \otimes \mathcal{I}^\psi \left(\frac{a_S}{a_W}, b_2 \right) \rightarrow \mathcal{I}^\psi \left(\frac{a_E}{a_N}, b_4 \right) \otimes \mathcal{I}^\psi \left(\frac{a_S}{a_E}, b_3 \right)$$

$$\langle \beta_1 | \otimes \langle \beta_2 | \mapsto \langle \beta_4 | \otimes \langle \beta_3 | \Omega(\beta_1, \beta_2, \beta_3, \beta_4, \mu_1, \mu_2)$$

The quasi-periodicity of the segment spaces and of Ω cancel out!

Complex CS as a Reshetikhin-Turaev invariant

Theorem [Me]

- $\mathcal{I}^\psi$ gives a well-defined invariant of tangles with decorated representations.
 - boundary points/segments get line bundles over space of colorings
 - a tangle is assigned a map between these line bundles
- for links we recover the usual complex Chern-Simons invariant

This is a funny version of Reshetikhin-Turaev:

- the objects are all 1-dimensional (this is an **invertible** theory)
- the space of *colors* is interesting and complicated (it's the tangle representation variety)
- to find natural bases of our modules we think of them as line bundles, not just vector spaces

Quantization

To get the new quantum hyperbolic invariant $\mathcal{Z}_N^\psi$, make these changes:

line bundles	→	rank N vector bundles
CS invariant of octahedron	→	braiding on quantum group modules
dilogarithms	→	quantum dilogarithms
$\mathcal{I}^\psi$	→	$\mathcal{Z}_N^\psi$

Segment modules are now

$$\mathcal{Z}_N^\psi(a, b) = \{f : \Lambda(b) \rightarrow \mathbb{C} \mid f(\beta + N) = af(\beta)\}$$

with basis vectors $\langle \beta |, \dots, \langle \beta + N - 1 |$ with $\langle \beta + N | = a^{-1} \langle \beta |$

and the braiding is now

$$\langle \beta_1 | \otimes \langle \beta_2 | \mapsto \sum_{n_3, n_4=0}^{N-1} \langle \beta_4 + n_4 | \otimes \langle \beta_3 + n_3 | \Omega_N(\beta_1, \beta_2, \beta_3 + n_3, \beta_4 + n_4, \mu_1, \mu_2)$$

where Ω_N is essentially Ω with dilogarithms replaced by quantum dilogarithms:

$$\Omega_N(\beta_i, \mu_j) = \frac{\varphi(\beta_4 - \beta_1)\varphi(\beta_2 - \beta_3 + \mu_2 - \mu_1)}{\varphi(\beta_2 - \beta_1 - \mu_1)\varphi(\beta_4 - \beta_3 + \mu_2)} \omega(\dots)$$

These matrix coefficients were found (not in this form) in joint work with N. Reshetikhin

Here we are using a version of Faddeev's quantum dilogarithm, with

$$\varphi(\zeta + 1) = \frac{\varphi(\zeta)}{1 - \omega^{\zeta+1}} \text{ so } \varphi(\zeta + N) = \frac{\varphi(\zeta)}{1 - \omega^{N\zeta}}$$

where $\omega^x = e^{2\pi i x/N}$. This is an analogue of dilogarithm dependence on branch choices:

$$z^1 = \frac{1}{1 - z} = (1 - \omega^{N\zeta})^{-1}$$

These factors cancel with the ones from our basis vectors, so we get a linear map independent of the choice of log-parameters.

If you have ever worked with invariants of this type you will have seen that keeping track of the logarithms is a pain! This avoids that.

Invariance and meridian logarithms

We can use the braidings above (and cups and caps) to interpret a tangle diagram as a tensor construction, which gives $\mathcal{Z}_N^\psi$.

Theorem [Me] $\mathcal{Z}_N^\psi(K, \rho, \mu)$ is invariant under colored Reidemeister moves, so does not depend on the diagram.

Why is this a vector bundle?

With ℓ the distinguished longitude eigenvalue, compare:

$$\mathcal{I}^\psi(K, \rho, \mu + 1) = \ell^2 \mathcal{I}^\psi(K, \rho, \mu)$$

$$\mathcal{Z}_N^\psi(K, \rho, \mu + N) = \ell^{-2} \mathcal{Z}_N^\psi(K, \rho, \mu)$$

There are now N independent choices of μ . Think of these as (cyclic) highest weights of modules.

$\mathcal{Z}_N^\psi(a, b)$ has a natural action of quantum torus $\mathcal{T}$

$$(Xf)(\beta) = f(\beta + 1)$$

$$(Yf)(\beta) = \omega^\beta f(\beta)$$

with $XY = \omega YX$.

Central character is $\chi(X^N) = a, \chi(Y^N) = b$.

Also want a central Z acting by ω^μ .

There is an algebra homomorphism $\eta : \mathcal{U}_\omega(\mathfrak{sl}_2) \rightarrow \mathcal{T}$

de Concini-Kac modules $V(\chi)$ arise from $\mathcal{T}$ -modules pulled back along η (not all, but generic ones)

Idea: this is a cluster presentation of $\mathcal{U}_\omega(\mathfrak{sl}_2)$

Now quasitriangular structure of $\mathcal{U}_\omega(\mathfrak{sl}_2)$ goes most of the way to proving Ω_N satisfies braid relations. Rest comes from relationship with $\mathcal{I}^\psi$.

What does “pullback” mean?

$$\begin{array}{ccc}
 \mathcal{U}_\omega(\mathfrak{sl}_2) & \xrightarrow{\eta} & \mathcal{T} \\
 & \searrow \pi\eta & \downarrow \pi \\
 & & \text{End}_{\mathbb{C}}\left(\mathcal{Z}_N^\psi(a, b)\right)
 \end{array}$$

Structure map π defines $x \cdot f = \pi(x)f$ for $x \in \mathcal{T}$

Thus if $u \in \mathcal{U}_\omega(\mathfrak{sl}_2)$ we have

$$u \cdot f = \pi(\eta(u))f$$

Explicitly the map is

$$\eta(K) = X$$

$$\eta(E) = \omega^{\frac{1}{2}} Y (Z - X)$$

$$\eta(F) = Y^{-1} (1 - Z^{-1} X^{-1})$$

$$\eta(\Omega) = \omega^{\frac{1}{2}} Z + \omega^{-\frac{1}{2}} Z^{-1}$$

Comes from cluster algebra of punctured disk

Here Z is related to the homology/puncture operators of [Bonahon and Liu, 2007]

Where did this come from?

- The original problem was to understand a quantum invariant of [Blanchet, Geer, Patureau-Mirand, and Reshetikhin, 2020]
- Because $\mathcal{U}_\omega(\mathfrak{sl}_2)$ does not have a universal R -matrix it is very hard to say what its braiding is or find matrix coefficients
- Reshetikhin had the idea to use $\mathcal{T}$ as on previous slide to solve for them
- Found Ω_N , up to an overall scalar
- My job was to figure out how to normalize our braiding. This involved a long detour through hyperbolic geometry. I did not know anything about complex Chern-Simons invariants when this started!

Future directions

- Perhaps these ideas can be used for other cyclic QHI constructions?
- Does this show a relationship with principal QHI?
- I claimed $\mathcal{Z}_N^\psi$ quantizes $\mathcal{I}^\psi$, but this is based on analogies. Can we prove this using geometric quantization?

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Where does the μ, λ behavior come from?

Classical

The octahedron function transforms as

$$\Omega(\mu_1 + 1, \dots) = \frac{b_1}{b_3} \Omega(\mu_1, \dots)$$

with corresponding rules for the other crossing signs and strands.

Multiplying the crossing contributions gives

$$\mathcal{I}^\psi(K, \rho, \mu + 1) = \ell^2 \mathcal{I}^\psi(K, \rho, \mu)$$

where ℓ is the distinguished longitude eigenvalue.

Quantum

For $\mathcal{Z}_N^\psi$ segment modules are only isomorphic if $\mu \equiv \mu' \pmod{N\mathbb{Z}}$

μ is a sort of (cyclic) highest weight

$$\mathcal{Z}_N^\psi(K, \rho, \mu + N) = \ell^{-2} \mathcal{Z}_N^\psi(K, \rho, \mu)$$

Boundary vector bundles

Given ρ with boundary eigenvalues m, ℓ we see that $\mathcal{Z}_N^\psi(K, \rho)$ is an element of the vector space

$$\mathcal{E}_{N(m, \ell)} = \{f : \Lambda(m) \rightarrow \mathbb{C} \mid f(\mu + N) = \ell^{-2} f(\mu)\}$$

It is natural to call these a vector bundle

$$\mathcal{Z}_N^\psi(T^2) \rightarrow \mathcal{R}(T^2)$$

over the representation variety of the boundary torus, and then

$$\mathcal{Z}_N^\psi(K) = \mathcal{Z}_N^\psi(S^3 \setminus \nu(K)) \text{ is a section of } \mathcal{Z}_N^\psi(T^2) = \mathcal{Z}_N^\psi(\partial(S^3 \setminus \nu(K)))$$

which is exactly what you'd expect from a TQFT